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A NEW CLASS OF ASYMPTOTICALLY UNIFORM MOTIONS OF A HEAVY SOLID BODY WITH A FIXED POINT*

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A class of asymptotically uniform motions of a heavy solid body /1/ is indicated, which has the property that for any distribution of the body mass it is always possible to select its initial position and initial angular velocity so that with infinitely increasing time the body motions approach asymptotically an unsteady uniform rotation. The set of such motions generally depends on one arbitrary constant.

1. The problem of motion of a heavy solid body with a fixed point reduces to the integration of a system of differential equations which determine in the system of coordinates accompanying the body the angular velocity vector $\boldsymbol{\omega}$ and the vector $\mathbf{v}$ of direction of the gravity force

$$A\boldsymbol{\omega}^{\cdot} = A\boldsymbol{\omega}\times\boldsymbol{\omega} + \Gamma(\mathbf{e}\times\mathbf{v}), \quad \mathbf{v}^{\cdot} = \mathbf{v}\times\boldsymbol{\omega} \tag{1.1}$$

where A is the tensor of body inertia at its fixed point, Γ is the maximum value of the force of gravity moment, and $\mathbf{e}$ is a unit vector directed from the fixed point to the center of mass. Equations (1.1) have the following first integrals:

$$\mathbf{v}\cdot\mathbf{v} = 1, \quad A\boldsymbol{\omega}\cdot\mathbf{v} = k, \quad A\boldsymbol{\omega}\cdot\boldsymbol{\omega} - 2\Gamma(\mathbf{e}\cdot\mathbf{v}) = 2E \tag{1.2}$$

The structure of solution presented here was obtained without restrictions on the distribution of mass in the body and is defined by the limit state of the body, i.e. it uniform rotation. Let $\boldsymbol{\omega}^0$ be the vector of angular velocity of uniform rotation. Then $\boldsymbol{\omega}^0 = \omega^0\mathbf{v}^0$, $\omega^{02}(A\mathbf{v}^0\times\mathbf{v}^0) + \Gamma(\mathbf{e}\times\mathbf{v}^0) = 0$. From this, using conventional methods we derive the equation of the Staude cone and the uniform rotation angular velocity

$$A\mathbf{v}^0\cdot(\mathbf{e}\times\mathbf{v}^0) = 0, \quad \omega^{02} = \frac{\Gamma[(A\mathbf{e}\cdot\mathbf{v}^0) - (\mathbf{e}\cdot\mathbf{v}^0)(A\mathbf{v}^0\cdot\mathbf{v}^0)]}{(A\mathbf{v}^0\cdot\mathbf{v}^0) - (A\mathbf{v}^0)^2} \tag{1.3}$$

The constants of integrals are calculated by formulas

$$\mathbf{v}^0\cdot\mathbf{v}^0 = 1, \quad k = \omega^0(A\mathbf{v}^0\cdot\mathbf{v}^0), \quad 2E = \omega^{02}(A\mathbf{v}^0\cdot\mathbf{v}^0) - 2\Gamma(\mathbf{e}\cdot\mathbf{v}^0)$$

We substitute for t the new variable

$$\tau = ce^{\lambda t} \tag{1.4}$$

where we set $\lambda < 0,\ c > 0$. Denoting differentiation with respect to τ by a prime, we rewrite Eqs.(1.1) as

$$\lambda\tau A\boldsymbol{\omega}' = A\boldsymbol{\omega}\times\boldsymbol{\omega} + \Gamma(\mathbf{e}\times\mathbf{v}), \quad \lambda\tau\mathbf{v}' = \mathbf{v}\times\boldsymbol{\omega} \tag{1.5}$$

where vectors $\boldsymbol{\omega}$ and $\mathbf{v}$ are defined by formulas

$$\boldsymbol{\omega} = \omega^0\mathbf{v}^0 + \boldsymbol{\Omega}, \quad \mathbf{v} = \mathbf{v}^0 + \boldsymbol{\gamma} \tag{1.6}$$

The substitution of (1.6) into (1.5) yields

$$\lambda\tau A\boldsymbol{\Omega}' = \omega^0(A\mathbf{v}^0\times\boldsymbol{\Omega} + A\boldsymbol{\Omega}\times\mathbf{v}^0) + \Gamma(\mathbf{e}\times\boldsymbol{\gamma}) + A\boldsymbol{\Omega}\times\boldsymbol{\Omega}, \qquad \lambda\tau\boldsymbol{\gamma}' = \mathbf{v}^0\times\boldsymbol{\Omega} + \omega^0(\boldsymbol{\gamma}\times\mathbf{v}^0) + \boldsymbol{\gamma}\times\boldsymbol{\Omega} \tag{1.7}$$

Let us show that system (1.7) admists a solution of the form of analytic functions of τ /2/

$$\boldsymbol{\Omega} = \sum_{m=1}^{\infty}\boldsymbol{\omega}_m\tau^m, \quad \boldsymbol{\gamma} = \sum_{n=1}^{\infty}\mathbf{v}_n\tau^n \tag{1.8}$$

The substitution of (1.8) into (1.7) yields

$$\lambda\sum_{m=1}^{\infty} mA\boldsymbol{\omega}_m\tau^m = \omega^0\Big(A\mathbf{v}^0\times\sum_{m=1}^{\infty}\boldsymbol{\omega}_m\tau^m\Big) - \omega^0\Big(\mathbf{v}^0\times\sum_{m=1}^{\infty}A\boldsymbol{\omega}_m\tau^m\Big) + \Gamma\Big(\mathbf{e}\times\sum_{n=1}^{\infty}\mathbf{v}_n\tau^n\Big) + \Big(\sum_{m=1}^{\infty}A\boldsymbol{\omega}_m\tau^m\times\sum_{m=1}^{\infty}\boldsymbol{\omega}_m\tau^m\Big) \tag{1.9}$$

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$$\lambda \sum_{n=1}^{\infty} n\mathbf{v}_n\tau^n = \mathbf{v}^0 \times \sum_{m=1}^{\infty} \boldsymbol{\omega}_m\tau^m - \omega^0\Big(\mathbf{v}^0 \times \sum_{n=1}^{\infty} \mathbf{v}_n\tau^n\Big) + \Big(\sum_{n=1}^{\infty} \mathbf{v}_n\tau^n \times \sum_{m=1}^{\infty} \boldsymbol{\omega}_m\tau^m\Big)$$

The coefficients of series (1.8) are successively obtained from equations

$$\lambda k A\boldsymbol{\omega}_k + \omega^0(\mathbf{v}^0 \times A\boldsymbol{\omega}_k) - \omega^0(A\mathbf{v}^0 \times \boldsymbol{\omega}_k) - \Gamma(\mathbf{e} \times \mathbf{v}_k) = (1 - \delta_{k1})\Big(\sum_{q=1}^{k-1} A\boldsymbol{\omega}_q \times \boldsymbol{\omega}_{k-q}\Big) \qquad (1.10)$$

$$\lambda k\mathbf{v}_k + \omega^0(\mathbf{v}^0 \times \mathbf{v}_k) - (\mathbf{v}^0 \times \boldsymbol{\omega}_k) = (1 - \delta_{k1})\Big(\sum_{q=1}^{k-1} \mathbf{v}_q \times \boldsymbol{\omega}_{k-q}\Big)$$

which follow from the requirement that relations (1.9) must be identities with respect to τ. By virtue of equality $\delta_{11} = 1$ vector components $\boldsymbol{\omega}_1, \mathbf{v}_1$ satisfy only the system of homogeneous equations in (1.10), hence the constant λ is defined as the zero of that system determinant

$$\begin{aligned}
&\Delta_1 = \lambda^2(a_0\lambda^4 + a_1\lambda^2 + a_2) = 0 \qquad (1.11)\\
&a_0 = A_1A_2A_3\\
&a_1 = \sum_{k=1}^{3}\{A_kA_{k+1}[\omega^{02}(A_k - A_{k+1})(\nu_k^0 - \nu_{k+1}^{02}) + \Gamma(e_k\nu_k^0 + e_{k+1}\nu_{k+1}^0)] +\\
&\quad A_k\omega^{02}\nu_k^{02}(A_k - A_{k+1} - A_{k+2})^2\}\\
&a_2 = \sum_{k=1}^{3}\{A_k[\omega^{02}(A_k - A_{k+1})(\nu_k^{02} - \nu_{k+1}^{02}) + \Gamma(e_k\nu_k^0 + e_{k+1}\nu_{k+1}^0)] \times\\
&\quad [\omega^{02}(A_{k+2} - A_k)(\nu_{k+2}^{02} - \nu_k^{02}) + \Gamma(e_k\nu_k^0 + e_{k+2}\nu_{k+2}^0)] +\\
&\quad 2\omega^{02}\nu_k^{02}\nu_{k+1}^0(A_k - A_{k+1} - A_{k+2})(A_k - A_{k+1} + A_{k+2}) \times\\
&\quad [\omega^{02}\nu_{k+1}^0(A_{k+1} - A_{k+2}) + \Gamma e_{k+1}] - A_k\nu_{k+1}^{02}[\omega^{02}\nu_{k+2}^0(A_{k+2} - A_k) +\\
&\quad \Gamma e_{k+2}]^2 + \omega^{02}\nu_k^{02}(A_k - A_{k+1} - A_{k+2})[\omega^{02}(A_{k+1} - A_{k+2})(\nu_{k+1}^{02} - \nu_{k+2}^{02}) +\\
&\quad \Gamma(e_{k+1}\nu_{k+1}^0 + e_{k+2}\nu_{k+2}^0)]\}
\end{aligned}$$

where A_i are the principal moments of inertia of the body, e_i are components of vector $\mathbf{e}$ in the principal coordinate axes of that tensor. Hence parameter λ in (1.4) is the root of the characteristic equation of the linear part of system (1.7). Since by virtue of the problem formulation $\lambda < 0$, Eq. (1.11) must have at least one negative root. Since it also admits a positive root, the considered here uniform rotation (1.3) is Liapunov unstable.

Parameters $A_i, e_i, \Gamma, \omega^0, \mathbf{v}^0$ must satisfy one of the following conditions: 1) $a_2 = 0,\ a_1 < 0$, 2) $a_2 < 0$, 3) $a_2 > 0,\ a_1 < 0,\ a_1^2 - 4a_2a_0 \geqslant 0$ /3/. In the first two cases there is among the roots of Eq. (1.11) one negative root, and in the third there are two negative roots. Note that the quantities $\boldsymbol{\omega}_1, \mathbf{v}_1$ are generally determined with an accuracy within one arbitrary constant.

Let us consider system (1.10) when $k > 1$ $(\delta_{k1} = 0)$. The determinant composed of coefficients at vector components $\boldsymbol{\omega}_k, \mathbf{v}_k$ is of the form $\Delta_k = \lambda^2k^2(a_0\lambda^4k^4 + a_1\lambda^2k^2 + a_2)$. On the assumption that $k > 1$ it vanishes only when $k^2 = a_2/(a_0\lambda^4)$. In case 1) and 2) we evidently have $\Delta_k \neq 0$. When the conditions of case 3) are satisfied, then when $\lambda = \lambda_1$ we have $k^2 = a_2/a_0\lambda_1^4$, and when $\lambda = \lambda_2$, $k^2 = a_0\lambda_1^4/a_2$, where λ_1, λ_2 denote the negative roots of Eq. (1.11), hence there must exist a $\lambda < 0$ such that $\Delta_k \neq 0$ when $k > 1$. Consequently, system (1.10) is always solvable for $\boldsymbol{\omega}_k, \mathbf{v}_k$ which are uniquely defined in terms of $\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_{k-1}$ and $\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}$, without the appearance of new arbitrary constants. The coefficients of series (1.8), thus, generally depend on one arbitrary constant.

Convergence of (1.8) is implies by Kamenkov's theorem /2/ on the basis of the property $\Delta_k \neq 0$ proved above in the case of $k > 1$. We denote by τ^* the convergence radius of series (1.8) and select the quantity c in (1.4) so that $c < \tau^*$. Then by virtue of (1.4), (1.6), and (1.8) motions of the body with unbounded increase of time asymptotically approach uniform rotation. Hence the following theorem is valid.

Theorem. The Euler-Poisson equations have in the case of arbitrary mass distribution in a heavy solid body a solution in the form of analytic functions of the variable $\tau = ce^{\lambda t}$

$$\boldsymbol{\omega} = \omega^0\mathbf{v}^0 + \sum_{m=1}^{\infty} \boldsymbol{\omega}_m\tau^m, \qquad \mathbf{v} = \mathbf{v}^0 + \sum_{n=1}^{\infty} \mathbf{v}_n\tau^n \qquad (1.12)$$

This solution defines the class of motions asymptotically approaching the unstable uniform rotation.

Note that up to the present asymptotic uniform motions were disclosed only in a small number of solutions (see the overview in /4/). Each of them was obtained for a fixed distribution of the body mass, and the asymptotically uniform motion of the body were defined for particular selection of input data. The solution derived in the present investigation holds for any arbitrary mass distribution, thus, not only unifying particular solutions for asymptotically uniform motions of the body in one class but, also, disclosing new cases of motions of a heavy solid body.

2. Let us consider the particular case of $\omega^0 = 0$. From (1.1) and (1.2) we obtain $\mathbf{v}^0 = -\mathbf{e}$, $k = 0$, $E = \Gamma$, and reduce (1.10) to the form

$$\mathbf{v}_k = \frac{1}{\lambda k}\Big[\boldsymbol{\omega}_k \times \mathbf{e} + (1 - \delta_{k1})\Big(\sum_{q=1}^{k-1} \mathbf{v}_q \times \boldsymbol{\omega}_{k-q}\Big)\Big] \qquad (2.1)$$

$$\lambda^2 k^2 A\boldsymbol{\omega}_k - \Gamma\boldsymbol{\omega}_k + \Gamma\mathbf{e}(\boldsymbol{\omega}_k \cdot \mathbf{e}) = (1 - \delta_{k1})\Big\{\lambda k\Big(\sum_{q=1}^{k-1} A\boldsymbol{\omega}_q \times \boldsymbol{\omega}_{k-q}\Big) + \Gamma\Big[\mathbf{e} \times \Big(\sum_{q=1}^{k-1} \mathbf{v}_q \times \boldsymbol{\omega}_{k-q}\Big)\Big]\Big\}$$

Coefficients of the characteristic equation (1.11) are:

$$a_0 = A_1A_2A_3,\ a_1 = -[A_1A_2(e_1^2 + e_2^2) + A_2A_3(e_2^2 + e_3^2) + A_1A_3(e_1^2 + e_3^2)]\,\Gamma$$

$$a_2 = (A_1e_1^2 + A_2e_2^2 + A_3e_3^2)\,\Gamma^2$$

Calculation of the discriminant of Eq.(1.11) yields

$$D = [e_1^4A_1^2(A_2 - A_3)^2 + e_2^4A_2^2(A_3 - A_1)^2 + e_3^4A_3^2(A_1 - A_2)^2 - 2e_1^2e_2^2A_1A_2(A_2 - A_3)(A_3 - A_1) - 2e_2^2e_3^2A_2A_3(A_3 - A_1)(A_1 - A_2) - 2e_1^2e_3^2A_1A_3(A_1 - A_2)(A_2 - A_3)]\,\Gamma^2$$

If among the principal moments of inertia there are equal ones, we have the evident inequality $D \geqslant 0$. If there are no equal moments of inertia, we can set $A_1 < A_2 < A_3$ and again obtain

$$D = \{[e_1A_1(A_2 - A_3) - e_2A_2(A_3 - A_1) - e_3A_3(A_1 - A_2)]^2 + 4e_2^2e_3^2(A_2 - A_1)(A_3 - A_1)\}\,\Gamma^2 \geqslant 0$$

i.e. Eq.(1.11) has four real roots for any distribution of the body mass.

When $\omega^0 = 0$ we obtain from (1.8) formulas (1.12) in which $\omega^0 = 0$ and the coefficients satisfy system (2.1). Analysis of the latter shows that terms of these series are generally determined with an accuracy to a single arbitrary constant, while for Lagrange and Hess gyroscopes there are two such constants. The results obtained in this Section enable us to formulate the following corollary of the theorem.

Corollary. In the case of arbitrary distribution of mass in a heavy solid body it is always possible to select its initial position and velocity so that, as $t \to \infty$ motion of the body will asymptotically approach the unsteady state of rest.

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REFERENCES

1. GORR G.V., The asymptotically uniform motions of a solid body with a fixed point in a potential force field. In: Mechanics of the Solid Body. Kiev, NAUKOVA DUMKA, Vol.12, 1980.
2. KAMENKOV G.V., Selected Works: Stability of Motion, Oscillations. Aerodynamics. Moscow, NAUKA, 1971.
3. RUMIANTSEV V.V., Stability of permanent rotation of a heavy solid body. PMM, Vol.20, No.1, 1956.
4. GORR G.V., KUDRIASHOVA L.V. and STEPANOVA L.A., Classic Problems of Solid Body Dynamics. Development and Current State. Kiev, NAUKOVA DUMKA, 1978.

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